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On the stability and uniqueness of the flow of a fluid through a porous medium

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Abstract. In this short note, we study the stability of flows of a fluid through porous media that satisfies a generalization of Brinkman's equation to include inertial effects. Such flows could have relevance to enhanced oil recovery and also to the flow of dense liquids through porous media. In any event, one cannot ignore the fact that flows through porous media are inherently unsteady, and thus, at least a part of the inertial term needs to be retained in many situations. We study the stability of the rest state and find it to be asymptotically stable. Next, we study the stability of a base flow and find that the flow is asymptotically stable, provided the base flow is sufficiently slow. Finally, we establish results concerning the uniqueness of the flow under appropriate conditions, and present some corresponding numerical results.

Mathematics Subject Classification. 76S05; 76E20.

Keywords. Brinkman model; Uniqueness; Stability of laminar flows.

1. Introduction

In this note, we shall study the stability of flows of a fluid that is governed by a generalization to Brinkman's equation that takes into account the effect of inertia. Brinkman [1,2] developed an equation for the flow of a fluid through a porous solid which reduces to the equation developed by Darcy [3] for the flow through a porous medium when one ignores the frictional effects within the fluid and to the equations governing Stokes flow when the effects of the friction at the pores are ignored. Forchheimer [4] suggested a modified "drag" due to the friction at the pore as he found the predictions of Darcy's equation to be not in keeping with experimental effects. The interaction term that he introduced leads to the equation becoming nonlinear. However, in the models proposed by Darcy, Brinkman and Forchheimer the non-linearity of the inertial effect is ignored. The justification offered by Darcy and Brinkman to ignore the effects of inertia is that the flow in a porous media is expected to be slow. However, as shown by Munaf et al. [5], inertial effects can become important in the flow of fluids through porous media under certain circumstances. In fact, in problems such as enhanced oil recovery where the oil is driven by steam at high pressure, when the pressure gradients are high or when the flow of dense fluids is considered, inertial effects can become important, or at least significant enough to be not ignored. In flows involving high pressures and high pressure gradients, it might be necessary to include the effect of the pressure on the viscosity as well as on the "drag" term that arises due to frictional effects at the pore.

Recently, Subramaniam and Rajagopal [6] investigated flow of fluids at high pressures and pressure gradients under the assumption that both the viscosity and the "drag coefficient" depend on the pressure. They found flow rates markedly different from those predicted by the classical model with constant viscosity and constant "drag coefficient." They also found the development of boundary layers (regions where the vorticity is much larger than the rest of the flow domain) wherein the high pressures are confined. Later, Kannan and Rajagopal [7] studied the flow of fluids through an inclined porous medium at high pressures and pressure gradients in the presence of the effects of gravity and their results show

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the development of boundary layers wherein the vorticity is concentrated. The flows considered by Subramaniam and Rajagopal [6] and Kannan and Rajagopal [7] are steady and of a special form such that the inertial term is identically zero. However, the flow fields considered in these and several other studies can be viewed only as approximations to the real flows that take place in a porous medium as the main assumption in such researches is that the flow is unidirectional. It is important to recognize that flows through porous media are inherently unsteady, and thus, one has to include at the very least the unsteady part of the inertial term. Moreover, flows through porous media are never truly one-dimensional as the flows take place through tortuous pores, and thus, one cannot neglect the nonlinear term in the inertia on that basis. In fact, when very high pressure gradients are involved the flow could be locally turbulent. Here, we shall not consider turbulent flows. We shall, however, modify Brinkman's equation to take into account the effects of inertia.

A detailed discussion of the various assumptions that go into the development of Brinkman's equation can be found in the recent article on a hierarchy of approximations for the flow of fluids through porous media by Rajagopal [8].[1] While discussing Darcy's approximation, Brinkman very astutely observed that "*Equation* (2.2), however, cannot be used as such. A first objection is that no viscous stress has been defined with relation to it. The viscous shearing stresses acting on a volume element of a fluid have been neglected; only the damping force of the porous mass η^{ν}/k has been retained. This is a good approximation for small permeabilities.[2]" When the permeability is large, it is necessary to include the effect of the viscous dissipation within the fluid in the modeling. Brinkman's equation can be derived systematically from the theory of mixtures (see Truesdell [9], Bowen [10], Atkin and Craine [11], Samohyl [12], Rajagopal and Tao [13] for a detailed discussion of the mechanics of mixtures) by making the following assumptions (see [8]):

1. The solid is a rigid porous solid, and thus, the balance of linear momentum of the solid can be ignored; the stresses in the solid are whatever they need to be to meet the balance of linear momentum of the solid.
2. Frictional effects between the fluid and the pore as well as frictional effects in the fluid due to the viscosity of the fluid are included.[3] The fluid will be assumed to be a linearly viscous fluid.
3. The flow is sufficiently slow that inertial effects in the fluid can be neglected.
4. The fluid density is assumed to be a constant.
5. The flow is steady.

We shall not enforce the requirement that inertial effects be neglected or that the flow be steady. Based on this generalization of the model due to Brinkman, we shall consider the stability of the base flow to finite disturbances and conditions under which we can establish its uniqueness. The seminal works of Reynolds [15] and Orr [16], followed by the work of Synge [17], Kampe de Feriet [18], Berker [19], Thomas [20], Hopf [21] laid the foundation to the stability of flows of Navier–Stokes fluids to finite disturbances and Serrin [22] built upon this work and was able to obtain conditions for the Reynolds number that would guarantee the stability of flows. In such a way he extended the work of Hopf and Thomas under which one could establish the uniqueness of flows of the Navier–Stokes fluid. We shall follow a similar procedure to establish the asymptotic stability of the base flow of a fluid that satisfies the equations developed by Brinkman and establish conditions under which the solution is unique. We show that the base flow is globally asymptotically stable, i.e., the L^2-norm of the disturbances to the basic flow decay exponentially in time, provided the base flow is sufficiently slow in the sense that the Reynolds number does not exceed a critical threshold. We are also able to establish that the base flow is unique under the same conditions.

[1] There are several obvious typographical errors which appear in the paper by Rajagopal [8]. The sign in front of in Eqs. (3.4), (3.7), (3.11), (3.14) and (4.8) should be a negative sign instead of a positive one.

[2] By Eq. (2.2) Brinkman is referring to Darcy's equation.

[3] A detailed discussion of the various interaction mechanisms between constituents of fluids can be found in the article by Johnson et al. [14].

Several mathematical studies concerning convection in porous media [23–28] have been carried out by coupling Brinkman's equation with the equation of balance of energy, the two equations coupled by a term accounting for the effects of buoyancy as in the Oberbeck–Boussinesq approximation [29–31]. Such a classic approximation is widely used but it is not an approximation that retains terms of like order in a perturbation (see the works by Rajagopal et al. for a detailed discussion of the issues [32] and a rigorous derivation of Oberbeck–Boussinesq-type approximations governing convection in a porous medium [33]). An up to date discussion of the literature pertinent to the stability of flows in porous media can be found in the recent book by Straughan [34].

The plan of the paper is as follows. In the next section, we document the governing equations and in Sect. 3, we study the asymptotic stability of the rest state. In the final section, we carry out the asymptotic stability analysis to a base flow and provide some corresponding numerical results.

2. Governing equations

The equation developed by Brinkman [1,2] is

$$-\nabla p + \mu\Delta \boldsymbol{v} - \alpha\boldsymbol{v} + \rho\boldsymbol{b} = \boldsymbol{0}. \tag{2.1}$$

In the above equation, μ denotes the fluid viscosity, α the drag coefficient due to the frictional resistance offered by the pore to the flow of the fluid, p the pressure, $\boldsymbol{v}$ the velocity and $\boldsymbol{b}$ the body force. We shall assume that both the viscosity and drag coefficient are positive. Since it is assumed that the fluid density ρ is constant, the fluid can only undergo isochoric motions, and thus, we have

$$\mathrm{div}\boldsymbol{v} = \boldsymbol{0}. \tag{2.2}$$

Equations (2.1) and (2.2) provide four scalar equations for the three components of the velocity and pressure. The above model due to Brinkman assumes that the flow is sufficiently slow that inertial effects in the fluid can be ignored. We shall consider a generalization that takes into account inertial effects due to the flow, namely

$$\rho\left[\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = -\nabla p + \mu\Delta\boldsymbol{v} - \alpha\boldsymbol{v} + \rho\boldsymbol{b}. \tag{2.3}$$

We notice that when α is zero, the above equation reduces to the Navier–Stokes equation.

We shall henceforth assume that the body force field is conservative with potential ϕ, i.e., $\boldsymbol{b} = -\nabla\phi$. Then, Eq. (2.3) can be rewritten as

$$\rho\left[\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = -\nabla P + \mu\Delta\boldsymbol{v} - \alpha\boldsymbol{v}, \tag{2.4}$$

where $P = p + \rho\phi$.

3. Uniqueness and stability in bounded domains

Let Ω be a bounded domain and let d denote its diameter. Let us non-dimensionalize Eqs. (2.4) and (2.2) according to

$$\boldsymbol{x}^* = \frac{\boldsymbol{x}}{d}, \quad \boldsymbol{v}^* = \frac{\boldsymbol{v}}{V}, \quad t^* = \frac{V}{d}t, \quad P^* = \frac{P}{\rho V^2}, \tag{3.1}$$

V being a reference velocity (henceforth, the maximum modulus of the velocity field will be taken as a reference value). By dropping the asterisks for simplicity of notation, Eqs. (2.4) and (2.2) become

$$\begin{cases} \mathscr{D}\mathscr{R}\left[\dfrac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = -\mathscr{D}\mathscr{R}\nabla P + \mathscr{D}\Delta\boldsymbol{v} - \boldsymbol{v}, \\ \mathrm{div}\boldsymbol{v} = 0, \end{cases} \tag{3.2}$$

where $\mathscr{R} = \rho V d/\mu$ and $\mathscr{D} = \mu/(\alpha d^2)$ are the Reynolds and Darcy numbers, respectively. Let $m_0 = (\bar{\boldsymbol{v}}, \bar{P})$ be a solution to (3.2) in Ω satisfying a Dirichlet-type boundary condition on $\partial\Omega$ and let us study its uniqueness and stability. We first introduce the perturbations $(\boldsymbol{u}, \Pi)$ to the basic solution m_0, i.e.,

$$\bar{\boldsymbol{v}} = \boldsymbol{v} + \boldsymbol{u}, \quad P = \bar{P} + \Pi, \tag{3.3}$$

and then we write down the evolution equations of the perturbations

$$\begin{cases} \mathscr{D}\mathscr{R}\left[\dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\bar{\boldsymbol{v}} + (\bar{\boldsymbol{v}}\cdot\nabla)\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right] & \\ \qquad = -\mathscr{D}\mathscr{R}\nabla\Pi + \mathscr{D}\Delta\boldsymbol{u} - \boldsymbol{u} & \text{in } \Omega\times]0,+\infty[, \\ \mathrm{div}\boldsymbol{u} = 0 & \text{in } \Omega\times]0,+\infty[, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega\times]0,+\infty[. \end{cases} \tag{3.4}$$

On forming the scalar product of $(3.4)_1$ with $\boldsymbol{u}$, integrating over the domain Ω and taking into account $(3.4)_2$, $(3.4)_3$ and that $\mathrm{div}\bar{\boldsymbol{v}} = 0$, we obtain

$$\mathscr{D}\mathscr{R}\frac{\mathrm{d}E}{\mathrm{d}t} = -2\mathcal{G}(\bar{\boldsymbol{v}}, \boldsymbol{u}, t)E(t), \tag{3.5}$$

where

$$E(t) = \|\boldsymbol{u}(\cdot,t)\|_2^2 = \int_\Omega |\boldsymbol{u}(\boldsymbol{x},t)|^2 \mathrm{d}V \tag{3.6}$$

is the kinetic energy associated with the perturbations, the functional $\mathcal{G}$ is defined as

$$\mathcal{G}(\bar{\boldsymbol{v}}, \boldsymbol{u}, t) = \frac{\|\boldsymbol{u}\|_2^2 + \mathscr{D}\left(\|\nabla\boldsymbol{u}\|_2^2 + \mathscr{R}\displaystyle\int_\Omega \boldsymbol{u}\cdot\bar{\mathbf{D}}\boldsymbol{u}\,\mathrm{d}V\right)}{\|\boldsymbol{u}\|_2^2}, \tag{3.7}$$

and

$$\bar{\mathbf{D}} = \frac{1}{2}\left[\nabla\bar{\boldsymbol{v}} + (\nabla\bar{\boldsymbol{v}})^T\right]. \tag{3.8}$$

Let $\lambda_i(\boldsymbol{x}, t)$ $(i = 1, 2, 3)$ be the eigenvalues of the symmetric second-order tensor $\bar{\mathbf{D}}(\boldsymbol{x}, t)$ and assume that

$$\lambda_{\min} = \inf_{t\geq 0}\min_{\boldsymbol{x}\in\Omega}\min\{\lambda_1(\boldsymbol{x},t), \lambda_2(\boldsymbol{x},t), \lambda_3(\boldsymbol{x},t)\} > -\infty. \tag{3.9}$$

(It is worth noting that, since $\mathrm{div}\bar{\boldsymbol{v}} = \mathrm{tr}\bar{\mathbf{D}} = 0$, $\lambda_{\min}$ is non-positive and $\lambda_{\min}$ vanishes if and only if the velocity field $\bar{\boldsymbol{v}}$ is constant in $\Omega \times [0, +\infty[$.) Then, the functional $\mathcal{G}$ defined through (3.7) is bounded from below in $\mathcal{I} \times [0, +\infty[$, $\mathcal{I}$ being the space of the kinematically admissible perturbations, that is, the space of divergence-free vector fields defined in Ω and vanishing on $\partial\Omega$. In fact, assumption (3.9) and the Poincaré inequality [35,36],

$$\|\nabla\boldsymbol{u}\|_2^2 \geq C(\Omega)\|\boldsymbol{u}\|_2^2 \quad \forall\boldsymbol{u}\in\mathcal{I}, \tag{3.10}$$

yield

$$\begin{aligned}\mathcal{G}(\bar{\boldsymbol{v}}, \boldsymbol{u}, t) &\geq \frac{\|\boldsymbol{u}\|_2^2 + \mathscr{D}\left(\|\nabla \boldsymbol{u}\|_2^2 - \mathscr{R}|\lambda_{\min}|\|\boldsymbol{u}\|_2^2\right)}{\|\boldsymbol{u}\|_2^2} \\ &\geq 1 + \mathscr{D}\left[C(\Omega) - \mathscr{R}|\lambda_{\min}|\right] \quad \forall(\boldsymbol{u}, t) \in \mathcal{I} \times [0, +\infty[.\end{aligned} \tag{3.11}$$

Next, by following similar arguments as in [37] one can prove that for all $t \in [0, +\infty[$ the functional $\mathcal{G}(\bar{\boldsymbol{v}}, \boldsymbol{u}, t)$ admits minimum in $\mathcal{I}$ and, in the light of (3.11),

$$\gamma \equiv \inf_{t \geq 0} \min_{\boldsymbol{u} \in \mathcal{I}} \mathcal{G}(\bar{\boldsymbol{v}}, \boldsymbol{u}, t) \geq 1 + \mathscr{D}\left[C(\Omega) - \mathscr{R}|\lambda_{\min}|\right]. \tag{3.12}$$

We are now in position to prove the following theorem.

Theorem 1. *Let $m_0 = (\bar{\boldsymbol{v}}, P)$ be a solution to (3.2) satisfying Dirichlet-type boundary conditions such that*

$$\gamma = \inf_{t \geq 0} \min_{\boldsymbol{u} \in \mathcal{I}} \mathcal{G}(\bar{\boldsymbol{v}}, \boldsymbol{u}, t) > 0, \tag{3.13}$$

with $\mathcal{G}$ as in (3.7). Then, m_0 is globally exponentially stable.

Proof. From (3.5) and (3.13) we deduce that

$$\frac{\mathrm{d}E}{\mathrm{d}t} \leq -\frac{2\gamma}{\mathscr{D}\mathscr{R}} E(t). \tag{3.14}$$

Integrating (3.14) yields

$$E(t) \leq E(0) \exp\left(-\frac{2\gamma t}{\mathscr{D}\mathscr{R}}\right), \tag{3.15}$$

and hence the kinetic energy associated with the perturbations decay exponentially in time. □

As a simple example of application of Theorem (1), if $\bar{\boldsymbol{v}} \equiv \mathbf{0}$, then $\gamma = 1 + \mathscr{D}C(\Omega) > 0$, and thus, the rest state is globally exponentially stable.

Another sufficient condition for the stability of the basic motion m_0 is given by the following corollary.

Corollary 1. *Let $m_0 = (\bar{\boldsymbol{v}}, P)$ be a solution to (3.2) satisfying Dirichlet-type boundary conditions on $\partial\Omega \times [0, +\infty[$ such that (3.9) holds. Assume that*

$$\mathscr{R} < \frac{1 + \mathscr{D}C(\Omega)}{\mathscr{D}|\lambda_{\min}|}. \tag{3.16}$$

Then, m_0 is globally exponentially stable.

Proof. The proof follows immediately from Theorem (1) and (3.12). □

It is worth noting that the stability condition (3.16) implies (3.13) but not vice versa. In addition, the stability condition (3.16) is easier to apply as it does not require the solution of a variational problem.

We conclude this section by remarking that if a solution to (3.2) under a prescribed initial condition on the velocity field meets the hypotheses of Theorem 1 or Corollary 1, then it is unique.

4. Laminar solutions

In this section, we are interested in the stability of laminar flows trough a porous medium that is bounded in only one direction. Then, once introduced a Cartesian frame of reference $Oxyz$ with basis $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, the porous layer may be represented by the domain $\Omega_d = \mathbb{R}^2 \times [0, d]$ and the laminar flows whose stability we shall investigate are of the form

$$\boldsymbol{v} = U(z)\boldsymbol{i}. \tag{4.1}$$

Following the procedure carried out in the previous section, we non-dimensionalize equations (2.4) and (2.2) according to (3.1) (in which d is now the thickness of the porous layer and $V = \max_{z\in[0,d]} |U(z)|$) to obtain (3.2) once again. It is easy to check that the following solutions to (3.2) represent all the possible laminar flows of the form (4.1):

$$\begin{cases} U(z) = c_1 \exp(\tau z) + c_2 \exp(-\tau z) + A_0, \\ P = \bar{P}(x) = -\dfrac{A_0}{\mathscr{D}\mathscr{R}}x + P_0, \end{cases} \tag{4.2}$$

where c_1, c_2, A_0 and P_0 are integration constants and $\tau = 1/\sqrt{\mathscr{D}}$.

As special cases of (4.2), for

- $U(0) = 0$, $U(1) = 1$ and $A_0 = 0$ one obtains the Couette-type flow

$$\begin{cases} U(z) = \dfrac{\sinh(\tau z)}{\sinh \tau}, \\ P = \bar{P}(x) = P_0, \end{cases} \tag{4.3}$$

- $U(0) = U(1) = 0$ and $A_0 \neq 0$ we get the Poiseuille-type flow[4]

$$\begin{cases} U(z) = \operatorname{sign}(A_0)\dfrac{\cosh\left(\dfrac{\tau}{2}\right) - \cosh\left[\tau\left(z - \dfrac{1}{2}\right)\right]}{\cosh\dfrac{\tau}{2} - 1}, \\ P = \bar{P}(x) = -\dfrac{A_0}{\mathscr{D}\mathscr{R}}x + P_0. \end{cases} \tag{4.4}$$

5. Stability of laminar flows

Let $\boldsymbol{u} = u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}$ and Π be the perturbations to the velocity and pressure fields given by (4.2), i.e.,

$$\boldsymbol{v} = U(z)\boldsymbol{i} + \boldsymbol{u}, \quad P = \bar{P}(x) + \Pi. \tag{5.1}$$

From (3.2), we deduce that the perturbations satisfy the following equations

$$\begin{cases} \mathscr{D}\mathscr{R}\left[\dfrac{\partial \boldsymbol{u}}{\partial t} + U\dfrac{\partial \boldsymbol{u}}{\partial x} + U' w \boldsymbol{i} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right] = -\mathscr{D}\mathscr{R}\nabla\Pi - \boldsymbol{u} + \mathscr{D}\Delta\boldsymbol{u}, \\ \operatorname{div}\boldsymbol{u} = 0, \end{cases} \tag{5.2}$$

the prime denoting differentiation with respect to z, and the boundary conditions

$$\boldsymbol{u} = \boldsymbol{0} \quad z = 0, 1. \tag{5.3}$$

From here on, we shall assume that the perturbations $\boldsymbol{u}$ and Π are periodic in the x and y directions with periods $2\pi/a_x$ and $2\pi/a_y$ ($a_x > 0$, $a_y > 0$), respectively, denote by Ω_p the periodicity cell

$$\Omega_p = \left[-\frac{\pi}{a_x}, \frac{\pi}{a_x}\right] \times \left[-\frac{\pi}{a_y}, \frac{\pi}{a_y}\right] \times [0, 1] \tag{5.4}$$

and let $a = \sqrt{a_x^2 + a_y^2}$ be the wave number.

[4] For the sake of brevity, we shall hereafter refer to the Couette-type and Poiseuille-type flows as "Couette" and "Poiseuille" flows.

5.1. Linear stability

On linearizing Eq. (5.2), we obtain

$$\begin{cases} \mathscr{D}\mathscr{R}\left[\dfrac{\partial \boldsymbol{u}}{\partial t}+U\dfrac{\partial \boldsymbol{u}}{\partial x}+U'w\boldsymbol{i}\right]=-\mathscr{D}\mathscr{R}\nabla\Pi-\boldsymbol{u}+\mathscr{D}\Delta\boldsymbol{u}, \\ \mathrm{div}\boldsymbol{u}=0. \end{cases} \tag{5.5}$$

By taking the third components of curl and curlcurl of $(5.5)_1$ and taking into account $(5.5)_2$, we deduce that the components of the perturbation to the velocity field may be found by solving the following boundary value problem

$$\begin{cases} \mathscr{D}\mathscr{R}\left(-\dfrac{\partial \Delta w}{\partial t}-U\dfrac{\partial \Delta w}{\partial x}+U''\dfrac{\partial w}{\partial x}\right)=\Delta w-\mathscr{D}\Delta\Delta w, \\ \mathscr{D}\mathscr{R}\left(\dfrac{\partial \zeta}{\partial t}+U\dfrac{\partial \zeta}{\partial x}-U'\dfrac{\partial w}{\partial y}\right)=-\zeta+\mathscr{D}\Delta\zeta, \\ \Delta_* u=-\dfrac{\partial^2 w}{\partial x\partial z}-\dfrac{\partial \zeta}{\partial y}, \\ \Delta_* v=-\dfrac{\partial^2 w}{\partial y\partial z}+\dfrac{\partial \zeta}{\partial x}, \\ w=\dfrac{\partial w}{\partial z}=0 \quad \text{on } z=0,1, \\ \zeta=0, \end{cases} \tag{5.6}$$

where $\zeta=\mathrm{curl}\boldsymbol{u}\cdot\boldsymbol{k}$ and

$$\Delta_*=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \tag{5.7}$$

is the two-dimensional Laplacian. From (5.6), we deduce that the only independent component of the perturbation to the velocity field is w as, once w is determined by solving Eq. $(5.6)_1$ subject to the boundary conditions $(5.6)_5$, all the other unknown scalar fields may be determined from the remaining equations and boundary conditions. Finally, as the perturbation to the pressure field is concerned, once all the components of $\boldsymbol{u}$ are determined, it may be determined by solving $(5.5)_1$.

Since the coefficients in $(5.6)_1$ depend only on z, Eq. $(5.6)_1$ admits solutions which depend on x, y and t exponentially. We consider therefore solutions of the form

$$w(x,y,z,t)=W(z)\exp[i(a_x x+a_y y-a_x ct)], \tag{5.8}$$

in which it is understood that the real parts of these expressions must be taken into consideration to obtain physically meaningful quantities. The wave speed c may be complex, i.e., $c=c_r+ic_i$, and the expression (5.8) thus represents waves which travel in the x and y coordinate directions with phase speed $a_x c_r/a$ and which grow or decay in time given by $\exp(-a_x c_i t)$. Such a wave is stable if $c_i>0$, unstable if $c_i<0$, and neutrally stable if $c_i=0$. If we now let $D=\mathrm{d}/\mathrm{d}z$, then on substituting the expression (5.8) into equation $(5.6)_1$ and boundary conditions $(5.6)_4$ we obtain the following boundary value problem[5]

$$\begin{cases} [\mathscr{D}(D^2-a^2)-1](D^2-a^2)W=ia_x\mathscr{D}\mathscr{R}[(U-c)(D^2-a^2)-U'']W, \\ W=DW=0 \quad \text{at } z=0,1. \end{cases} \tag{5.9}$$

The fourth-order system (5.9) can be solved by using the Chebyshev tau method [38], which is a spectral technique coupled with the QZ algorithm. For Poiseuille flow, the critical thresholds for the

[5] Equation $(5.9)_1$ represents the generalization of the Orr–Sommerfeld equation to laminar flows in a porous medium.

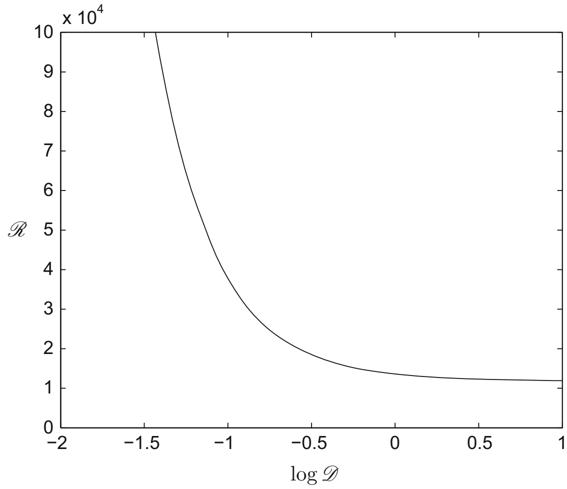

Fig. 1. Visual representation of the Poiseuille flow linear instability thresholds with critical Reynolds number $\mathscr{R}$ plotted against $\log \mathscr{D}$

Reynolds number shown in Fig. 1 correspond to comparable studies on Brinkman flow by Hill and Straughan [39]. On the contrary, Couette flow does not yield instability thresholds utilizing linear theory.

5.2. Nonlinear stability

In order to study the nonlinear stability of the laminar flows (4.2), we follow the same arguments as in Sect. 3 but modifying the notations slightly. More precisely, we introduce the functional

$$\mathcal{F}(U, \boldsymbol{u}) \equiv \frac{\|\boldsymbol{u}\|_2^2 + \mathscr{D}\left(\|\nabla \boldsymbol{u}\|_2^2 + \mathscr{R}\displaystyle\int_{\Omega_p} U' u w \mathrm{d}V\right)}{\|\boldsymbol{u}\|_2^2}, \tag{5.10}$$

and set

$$\gamma(a_x, a_y) \equiv \min_{\boldsymbol{u} \in \mathcal{I}_p} \mathcal{F}(U, \boldsymbol{u}), \tag{5.11}$$

where the space of the kinematically admissible perturbations $\mathcal{I}_p$ is the space of the divergence-free vector fields $\boldsymbol{u}$ defined in Ω_p such that

$$\begin{cases} \boldsymbol{u}\left(-\dfrac{\pi}{a_x}, y, z\right) = \boldsymbol{u}\left(\dfrac{\pi}{a_x}, y, z\right) & \forall (y,z) \in \left[-\dfrac{\pi}{a_y}, \dfrac{\pi}{a_y}\right] \times [0,1], \\ \boldsymbol{u}\left(x, -\dfrac{\pi}{a_y}, z\right) = \boldsymbol{u}\left(x, \dfrac{\pi}{a_y}, z\right) & \forall (x,z) \in \left[-\dfrac{\pi}{a_x}, \dfrac{\pi}{a_x}\right] \times [0,1], \\ \boldsymbol{u}\,(x, y, 0) = \boldsymbol{u}\,(x, y, 1) = \boldsymbol{0} & \forall (x,y) \in \left[-\dfrac{\pi}{a_x}, \dfrac{\pi}{a_x}\right] \times \left[-\dfrac{\pi}{a_y}, \dfrac{\pi}{a_y}\right]. \end{cases} \tag{5.12}$$

In this way, we may state that if $\gamma_p(a_x, a_y) > 0$, then the laminar flow (4.2) is non-linearly exponentially stable with respect to all perturbations periodic in the x and y directions with periods $2\pi/a_x$ and $2\pi/a_y$ as

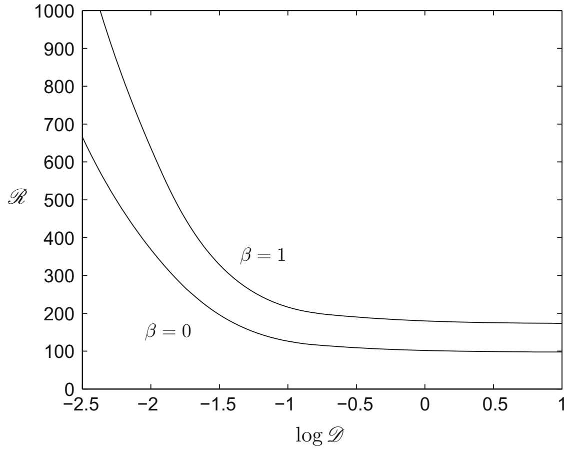

Fig. 2. Visual representation of the Poiseuille flow nonlinear stability thresholds with critical Reynolds number $\mathscr{R}$ plotted against $\log \mathscr{D}$. The thresholds for β values between 0 and 1 are contained between the $\beta = 0$ and $\beta = 1$ lines

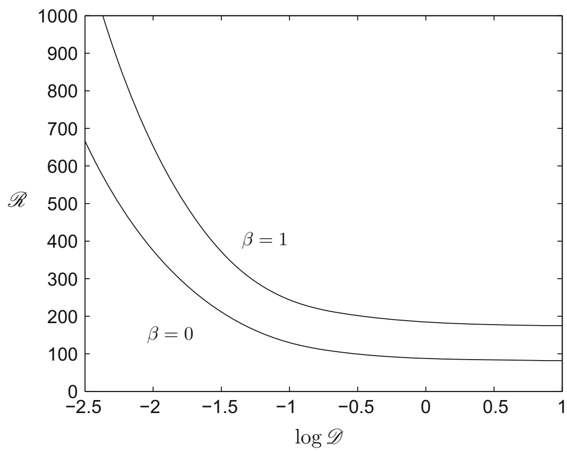

Fig. 3. Visual representation of the Couette flow nonlinear stability thresholds with critical Reynolds number $\mathscr{R}$ plotted against $\log \mathscr{D}$. The thresholds for β values between 0 and 1 are contained between the $\beta = 0$ and $\beta = 1$ lines

$$\|\boldsymbol{u}(\cdot,t)\|_2^2 \leq \|\boldsymbol{u}(\cdot,0)\|_2^2 \exp\left[-\frac{2\gamma_p(a_x,a_y)}{\mathscr{D}\mathscr{R}}t\right] \quad \forall \boldsymbol{u} \in \mathcal{I}_p. \tag{5.13}$$

In conclusion, from (5.13) we may state the following theorem.

Theorem 2. *Assume that*

$$\gamma_{\rm cr} \equiv \min_{a_x,a_y>0} \gamma_p(a_x,a_y) > 0. \tag{5.14}$$

Then the laminar flow (4.2) *is globally exponentially stable.*

The Euler–Lagrange equations corresponding to the variational problem (5.11) are

$$\begin{cases} \nabla\chi + (1-\sigma)\boldsymbol{u} - \mathscr{D}\Delta\boldsymbol{u} + \dfrac{1}{2}\mathscr{D}\mathscr{R}U'(w\boldsymbol{i} + u\boldsymbol{k}) = \boldsymbol{0}, \\ \mathrm{div}\boldsymbol{u} = 0, \end{cases} \tag{5.15}$$

where χ is a Lagrange multiplier associated with the incompressibility constraint. Then, the number $\gamma_p(a_x, a_y)$ is the least eigenvalue σ of the characteristic value problem (5.15)and (5.12).

Since the Euler–Lagrange equations (5.15) are linear, we may follow the same arguments as those employed for the linear stability analysis. More specifically, we take the third components of curl and curlcurl of $(5.15)_1$, use $(5.15)_2$ to get $(5.6)_3$, $(5.6)_4$ once again and look for solutions of the form

$$\begin{cases} w(x,y,z) = W(z)\exp[i(a_x x + a_y y)], \\ \zeta(x,y,z) = \mathrm{curl}\boldsymbol{u}\cdot\boldsymbol{k} = \Psi(z)\exp[i(a_x x + a_y y)] \end{cases} \tag{5.16}$$

to reduce the eigenvalue problem (5.15) and (5.12) to

$$\begin{cases} \mathscr{D}(D^2-a^2)^2 W + (\sigma-1)(D^2-a^2)W \\ \qquad\qquad + \dfrac{\mathscr{D}\mathscr{R}}{2}\left(2ia_x U' DW + ia_y U'\Psi + ia_x U'' W\right) = 0, \\ \mathscr{D}(D^2-a^2)\Psi + (\sigma-1)\Psi + \dfrac{\mathscr{D}\mathscr{R}}{2} ia_y U' W = 0, \\ W = DW = \Psi = 0 \quad \text{at } z = 0, 1. \end{cases} \tag{5.17}$$

The sixth-order system (5.17) has been solved using the Chebyshev tau method [38] for Poiseuille (Fig. 2) and Couette (Fig. 3) flows. We let $a_x = a\sqrt{\beta}$ and $a_y = a\sqrt{1-\beta}$, such that $\beta \in [0,\, 1]$ for the range of a_x and a_y values which comprise wavenumber a.

The numerical results for Poiseuille flow are in good agreement with those in [39].

Although there is some quantitative differences with the results for Poiseuille flow, the Couette flow nonlinear stability thresholds have a similar structure.

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